2935. [2004 : 174] *Proposed by Titu Zvonaru, Bucharest, Romania.*

Suppose that a, b, and c are positive real numbers which satisfy $a^2+b^2+c^2=1$, and that $n>1$ is a positive integer. Prove that

$$\frac{a}{1-a^n}+\frac{b}{1-b^n}+\frac{c}{1-c^n} \;\geq\; \frac{(n+1)^{1+\frac{1}{n}}}{n}\,.$$

Solution by Arkady Alt, San Jose, CA, USA.

For $0<x<1$ we have, by the AM–GM Inequality,

$$\begin{aligned}\big(x(1-x^n)\big)^n &= \frac{nx^n(1-x^n)^n}{n} \;\leq\; \frac{1}{n}\left(\frac{nx^n+n(1-x^n)}{n+1}\right)^{n+1}\\ &= \frac{n^n}{(n+1)^{n+1}}\,,\end{aligned}$$

from which we see that $x(1-x^n)\leq \dfrac{n}{(n+1)^{1+\frac{1}{n}}}$. Hence,

$$\begin{aligned}\sum_{\text{cyclic}}\frac{a}{1-a^n} &= \sum_{\text{cyclic}}\frac{a^2}{a(1-a^n)} \;\geq\; \sum_{\text{cyclic}}\frac{a^2(n+1)^{1+\frac{1}{n}}}{n}\\ &= \frac{(n+1)^{1+\frac{1}{n}}}{n}(a^2+b^2+c^2) \;=\; \frac{(n+1)^{1+\frac{1}{n}}}{n}\,.\end{aligned}$$

Equality holds when $n=2$ and $a=b=c=1/\sqrt{3}$.

Also solved by MICHEL BATAILLE, Rouen, France; JOSÉ LUIS DÍAZ-BARRERO, Universitat Politècnica de Catalunya, Barcelona, Spain; JOHN G. HEUVER, Grande Prairie, AB; JOE HOWARD, Portales, NM, USA; WALTHER JANOUS, Ursulinengymnasium, Innsbruck, Austria; D. KIPP JOHNSON, Beaverton, OR, USA; PHIL McCARTNEY, Northern Kentucky University, Highland Heights, KY, USA; BABIS STERGIOU, Chalkida, Greece; LI ZHOU, Polk Community College, Winter Haven, FL, USA; and the proposer.

Both Howard and the proposer remarked that this problem is a generalization of Crux 2738 [2002 : 180; 2003 : 243]. *Heuver remarked that Crux 1445* [1989 : 148; 1990 : 216] *by the late Murray Klamkin and Andy Liu dealt with a more generalized version of this problem. Janous, noticing that the lower bound is not sharp, offered three conjectures, one of which is as follows:* Let x_1, x_2, $\ldots$, x_n be positive real numbers satisfying $x_1^2+x_2^2+\cdots+x_n^2=1$. Then, for all $p>0$, we have $\displaystyle\sum_{j=1}^{n}\frac{x_j}{1-x_j^p}\geq\frac{n^{(p+1)/2}}{n^{p/2}-1}$.

Though a few solvers stated that equality holds in the given inequality only if $n=2$ *and* $a=b=c=1/\sqrt{3}$, *no one actually gave a detailed proof (though this is not difficult). Indeed, from the proof given in the solution above we see that if equality holds, then we must have* $nx^n=1-x^n$, *or* $x=\dfrac{1}{\sqrt[n]{n+1}}$ *which implies that* $a=b=c=\dfrac{1}{(n+1)^{1/n}}$. *From* $a^2+b^2+c^2=1$, *we then get* $\dfrac{3}{(n+1)^{2/n}}=1$ *or* $(n+1)^2=3^n$, *which clearly holds when* $n=2$. *But by a simple induction, one can easily show that* $3^n>(n+1)^2$ *for all* $n\geq 3$, *and the conclusion follows.*